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'LETTERS' SECTION

The Frame of Reference in Relativistic Theories

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The usual identification of the physical frame of reference with a special coordinate system is not satisfactory for several reasons (Bunge, 1966). The most important one is that in this case the observable quantities of every relativistic theory can not be given in a general covariant manner with regard to any substitution of the space-time coordinates x^μ. But this is necessary in the framework of a general theory of measurements, which should be free from any coordinate effects. In doing this, it is not allowed to connect the frame of reference with a special coordinate system. Therefore we will define the frame of reference in the following general covariant way.

(i) We introduce a 'vector field of reference' u^μ, which is the tangent field of non-crossing time-like world-lines. Such a 'congruence of curves' describes the history of physical observers or of a laboratory in the space-time. Its parameter representation is

$$x^\alpha_{(i)} = x^\alpha(c^\alpha_{(i)}, s_{(i)}) \tag{1}$$

Herein (i) signifies the ith world-line, the three numbers $c^a_{(i)}$ $(a = 1,2,3)$ determine the intersection point of the ith world-line with the three-dimensional local space-like hypersurface orthogonal to the world-lines (laboratory or rest space of the observers), and $s_{(i)}$ signifies the length of the ith world-line (eigentime of the observers). Thus the field of reference is given by

$$u^\alpha = \frac{\partial x^\alpha_{(i)}}{\partial s_{(i)}} \qquad \text{with } u^\alpha u_\alpha = 1 \tag{2}$$

(ii) The observable quantities of any relativistic theory are defined with regard to this field of reference by projections upon the local 'time-axis' u^μ and onto the local rest space of the observers (laboratory) orthogonal to u^μ (Synge, 1960). For this last projection the projection tensor

$$h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu \tag{3a}$$

exists with the properties

$$h_{\mu\nu} = h_{\nu\mu}, \qquad h_{\mu\alpha} h_\nu{}^\alpha = h_{\mu\nu}, \qquad h_{\mu\nu} u^\nu = 0, \qquad h_\mu{}^\mu = 3 \tag{3b}$$

where $g_{\mu\nu}$ is the metric tensor of the space-time.

If, for instance, $T_{\mu\nu}$ is the energy-momentum tensor of any relativistic theory, then

$$\rho = T_{\mu\nu} u^\mu u^\nu \tag{4a}$$

is the observable energy-density,

$$p_\mu = h_\mu{}^\alpha T_{\alpha\beta} u^\beta \tag{4b}$$

the space-like 3-momentum-density ($p_\mu u^\mu = 0$),

$$\theta_{\mu\nu} = h_\mu{}^\alpha h_\nu{}^\beta T_{\alpha\beta} \tag{4c}$$

the space-like three-dimensional stresses ($\theta_{\mu\nu} u^\nu = 0$) and

$$p = -\tfrac{1}{3}\theta_\mu{}^\mu \tag{4d}$$

the pressure. In the same covariant way the other quantities of the theory decay into observables.

(iii) Such a field of reference is characterized by its kinematical properties. The space-like 'distance-vector' of the ith world-line $x^\alpha_{(i)}$ to any infinitesimally adjacent 'central-line' $x^\alpha_{(0)}$ is, according to (1),

$$r^\alpha_{(i)} = h_\beta{}^\alpha \delta x^\beta_{(i)} \tag{5a}$$

with

$$\delta x^\beta_{(i)} = \frac{\partial x^\beta_{(i)}}{\partial c^\alpha_{(i)}} \delta c^\alpha_{(i)}, \qquad \delta c^\alpha_{(i)} = c^\alpha_{(i)} - c^\alpha_{(0)} \tag{5b}$$

Its change with time is†

$$v^\alpha_{(i)} = h_\beta{}^\alpha r^\beta_{(i)||\nu} u^\nu = (\omega_\beta{}^\alpha + \sigma_\beta{}^\alpha) r^\beta_{(i)} + \tfrac{1}{3}\vartheta r^\alpha_{(i)} \tag{6a}$$

with

$$\left.\begin{aligned} \omega_{\alpha\beta} &= \tfrac{1}{2}(u_{\mu||\nu} - u_{\nu||\mu}) h_\alpha{}^\mu h_\beta{}^\nu \\ \sigma_{\alpha\beta} &= \tfrac{1}{2}(u_{\mu||\nu} + u_{\nu||\mu}) h_\alpha{}^\mu h_\beta{}^\nu - \tfrac{1}{3}\vartheta h_{\alpha\beta} \\ \vartheta &= u^\mu_{||\mu} \end{aligned}\right\} \tag{6b}$$

Herein $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ signifies the rigid rotation of the field of reference with the angular velocity $\omega = \sqrt{(\frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta})}$, $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$ ($\sigma_\alpha{}^\alpha = 0$) determines its shear, and ϑ is its isotropic expansion (Ehlers, 1961).

Besides these internal kinematical properties with regard to the central line, we have as an external property, the acceleration of the reference system along the central line:

$$\dot{u}_\mu = u_{\mu||\nu} u^\nu \tag{6c}$$

(6b) and (6c) can be combined to form the gradient of u_μ:

$$u_{\mu||\nu} = \omega_{\mu\nu} + \sigma_{\mu\nu} + \tfrac{1}{3}\vartheta h_{\mu\nu} + \dot{u}_\mu u_\nu \tag{7}$$

† $||\nu$ signifies the covariant derivative with respect to the coordinate x^ν.

(iv) All these kinematical quantities can be determined in principle by suitable experiments, since they cause 'forces' in the rest-space of the observers. Thus $\omega_{\mu\nu}$ produces 'Coriolis-forces' and $\dot{u}_\mu$ results in 'gravitational forces' (Hönl &Dehnen, 1966). Therefore, $\omega_{\mu\nu}$ can be determined with a gyroscope and $\dot{u}_\mu$ with a spring balance and masses. For the determination of $\sigma_{\mu\nu}$ and ϑ, optical experiments are suitable, for instance the measurement of frequency-shifts or interference experiments.

(v) Fields of reference with

$$u_{\mu||\nu} = 0 \Leftrightarrow \begin{matrix} \omega_{\mu\nu} = 0, \ \sigma_{\mu\nu} = 0 \\ \vartheta = 0, \ \ \dot{u}_\mu = 0 \end{matrix} \tag{8}$$

represent the inertial systems whereas with the weaker condition

$$\sigma_{\mu\nu} = 0, \qquad \vartheta = 0 \tag{9}$$

they are the rigid frames of reference.

(vi) Every congruence of time-like world-lines is available as a field of reference. But the kinematical properties of such a field of reference can not be stated in any way, since the conditions of integrability of (7) (with given $\omega_{\mu\nu}$, $\sigma_{\mu\nu}$, ϑ and $\dot{u}_\mu$) are conditions for the structure of the Riemann tensor of space-time. Thus the gravitation has an important influence on the kinematical properties of the fields of reference. For instance, in vacuum ($R_{\mu\nu} = 0$) inertial systems (8) exist only, if the space-time is flat ($R_{\alpha\beta\gamma\delta} = 0$).

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